Econ 211

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Expected Utility: The Classic Theory

Motivating Example

- Suppose you are on the last round of the TV show *Who Wants to be a Millionaire*?
- You have narrowed down to two possible answers
  - Guess wrong: go home with $32,000
  - Guess right: go home with $1,000,000
- Walk away: go home with $500,000 for certain
- What do you do?

Gambles

- We need a way to make choices between uncertain options, eg gambles
- Consider a gamble called A, for example
  - Possible outcomes are indexed by $i = 1, 2, 3, \ldots, n$
  - Probability of outcome i: p_i
  - Value of outcome i: x_i
  - Gamble is them summarized by $(p_1, x_1; p_2, x_2; \ldots; p_n, x_n)$
- Examples:
  - Guess from Millionaire example: $\left(\frac{1}{2}, \$32000; \frac{1}{2}, \$1000000\right)$
  - Walk away: $(1, \$500000)$
  - Roll die, get paid the amount of the roll in dollars: $\left(\frac{1}{6}, \$1; \frac{1}{6}, \$2; \frac{1}{6}, \$3; \frac{1}{6}, \$4; \frac{1}{6}, \$5; \frac{1}{6}, \$6\right)$

Expected Value

- Expected value of gamble A:

$$EV(A) = \sum_{i}^{n} p_i x_i = p_1 x_1 + p_2 x_2 + \ldots + p_n x_n$$

- Examples:
  - Guess from Millionaire: $\frac{1}{2}$ \$1,000,000 $+\frac{1}{2}$ \$32,000 $=$ \$516,000
  - Die roll: $\frac{1}{6}\$1 + \frac{1}{6}\$2 + \frac{1}{6}\$3 + \frac{1}{6}\$4 + \frac{1}{6}\$5 + \frac{1}{6}\$6 = \$3.50$

Expected Utility

- Expected utility
  - Consumer assigns utility $u(x)$ to wealth x
  - Expect utility theory says that

$$EU(A) = \sum_{i}^{n} p_i u(x_i) = p_1 u(x_1) + p_2 u(x_2) + \ldots + p_n u(x_n)$$

- Consumers will choose the gamble that maximizes expected utility

What Shape Should $u(x)$ Have?

- Consider the following game: I will flip a coin until the first heads comes up. If the first heads is on flip number n, then I'll pay you $\$2^n$. How much would you pay to play this game?
  - Originally proposed by Bernoulli (1738, reprinted 1954)
  - Known as the *St. Petersberg Paradox*
- What is the expected value of this game?
  - $EV = \frac{1}{2}2 + \frac{1}{4}4 + \frac{1}{8}8 + \ldots = 1 + 1 + 1 + \ldots = \infty$
- It is clear that there is a *diminishing marginal utility of money*
  - Intuition: an extra \$1000 is massive windfall for a very poor person but not even noticeable for very rich person
- Means that $u(x)$ is concave, which represents *risk-averse* preferences
  - Can also have *risk-seeking* preferences (convex $u(x)$) or *risk-neutral* preference (linear $u(x)$)

Risk Aversion

- One possible family of functions: $u(x) = x^{\alpha}$
- Example: $u(x) = \sqrt{x}$, ie $\alpha = \frac{1}{2}$
  - Expected utility of \$9 for certain?

$$EU(1, \$9) = 1 \cdot u(\$9) = \sqrt{9} = 3$$

  - Expected utility of a fair coin flip for \$25?

$$EU\left(\frac{1}{2}, \$25; \frac{1}{2}, \$0\right) = \frac{1}{2}u(\$25) + \frac{1}{2}u(\$0) = \frac{1}{2} \cdot 5 + \frac{1}{2} \cdot 0 = 2.5$$

  - Would decision-maker prefer \$9 for certain or a coin flip for \$25? certain amount, even though coin flip has expected payoff of $\$12.50 > \$9.00$

Lab Evidence

- Subjects: 175 university students
- Choose either option A or B in *each* row:

TABLE 1—THE TEN PAIRED LOTTERY-CHOICE DECISIONS WITH LOW PAYOFFS

Option A	Option B	Expected payoff difference
1/10 of \$2.00, 9/10 of \$1.60	1/10 of \$3.85, 9/10 of \$0.10	\$1.17
2/10 of \$2.00, 8/10 of \$1.60	2/10 of \$3.85, 8/10 of \$0.10	\$0.83
3/10 of \$2.00, 7/10 of \$1.60	3/10 of \$3.85, 7/10 of \$0.10	\$0.50
4/10 of \$2.00, 6/10 of \$1.60	4/10 of \$3.85, 6/10 of \$0.10	\$0.16
5/10 of \$2.00, 5/10 of \$1.60	5/10 of \$3.85, 5/10 of \$0.10	−\$0.18
6/10 of \$2.00, 4/10 of \$1.60	6/10 of \$3.85, 4/10 of \$0.10	−\$0.51
7/10 of \$2.00, 3/10 of \$1.60	7/10 of \$3.85, 3/10 of \$0.10	−\$0.85
8/10 of \$2.00, 2/10 of \$1.60	8/10 of \$3.85, 2/10 of \$0.10	−\$1.18
9/10 of \$2.00, 1/10 of \$1.60	9/10 of \$3.85, 1/10 of \$0.10	−\$1.52
10/10 of \$2.00, 0/10 of \$1.60	10/10 of \$3.85, 0/10 of \$0.10	−\$1.85

- Repeated for 20x, 50x, 90x payoffs

Source: Holt and Laury (2002)

Expected Results

- How should responses change as subject progresses through price list from top to bottom?
  - Note that option B is always riskier than option A
  - Should prefer option A at top of price list
  - By bottom row, should switch to preferring option B
- Where do you switch if risk-neutral? switch from A to B after row 4
- What if risk-averse? switch farther down list
- What if risk-seeking? switch farther up list
- How should responses change with stakes? Three possibilities:
  1. Constant relative risk aversion: choices between options A and B should not depend on stakes
  2. Increasing relative risk aversion: choices are *more* risk averse as stakes go up (i.e. switch later)
  3. Decreasing relative risk aversion: choices are less *risk* averse as stakes go up (i.e. switch earlier)

Results: Holt and Laury

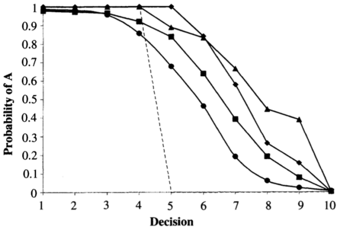


FIGURE 2. PROPORTION OF SAFE CHOICES IN EACH DECISION: DATA AVERAGES AND PREDICTIONS

Note: Data averages for low real payoffs [solid line with dots], 20x real [squares], 50x real [diamonds], 90x real payoffs [triangles], and risk-neutral prediction [dashed line].

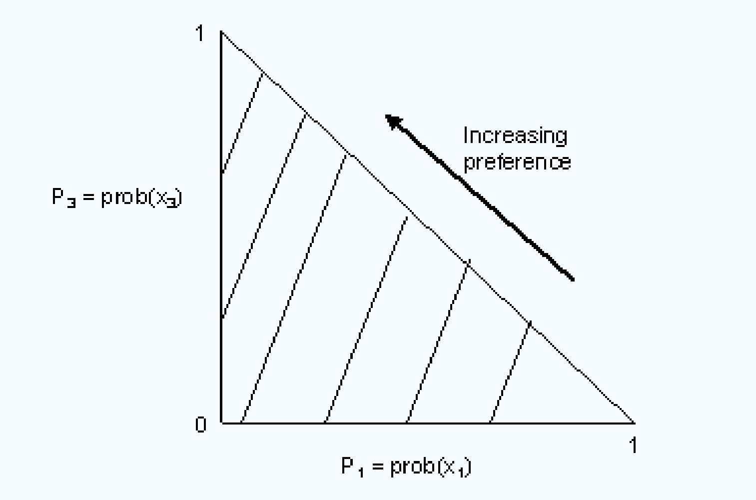
- Is the average participant risk averse, risk neutral, or risk loving?
  - Risk averse: note average switch point is well past row 5
- What is type of relative risk aversion?
  - Increasing relative risk aversion: note lines move out as stakes increase

Source: Holt and Laury (2002)

Machina Triangles

- How do we graph risky prospects themselves?
- Suppose we fix payoff amounts $x_1 < x_2 < x_3$
- Let p_1, p_2, and p_3 vary
- Since $p_1 + p_2 + p_3 = 1$, really just two degrees of freedom
- Put p_1 on horizontal axis and p_3 on vertical axis
- Possible gambles lie in the triangle defined by $p_1 \geq 0$, $p_3 \geq 0$, and $p_1 + p_3 \leq 1$, hence the name *Machina triangle*
- Any gamble can be represented at a point on this graph:
  - x_1 for certain: $(1, 0)$
  - x_2 for certain: $(0, 0)$
  - x_3 for certain: $(0, 1)$
  - x_1 and x_2 with equal probability: $(\frac{1}{2}, 0)$
  - x_1, x_2, and x_3 with equal probability: $(\frac{1}{3}, \frac{1}{3})$

Machina Triangle



Expected Utility in the Machina Triangle

- What do indifference curves in the Machina triangle look like for EUT?
- Set $EU = K$:

$$p_1 u(x_1) + (1 - p_1 - p_3) u(x_2) + p_3 u(x_3) = K$$

- Solve for p_3:

$$p_3 = \frac{u(x_2) - u(x_1)}{u(x_3) - u(x_2)} p_1 + C$$

- Indifference curves on Machina triangle are straight parallel lines with positive slope (increasing preference up and to the left)
- More risk aversion: steeper slope

Violations of Expected Utility Theory

The Allais Paradox: Version 1

1. Choose your preferred option:
   A: Receive \$100 million for certain
   B: 10% chance of \$500 million, 89% chance of \$100 million, 1% chance of no money
2. Choose your preferred option:
   A': 11% chance of \$100 million, 89% chance of no money
   B': 10% chance of \$500 million, 90% chance of no money

- Typical choice pattern? $A \succeq B$; $B' \succeq A'$

- $EU(A) = u(100)$
- $EU(B) = .1u(500) + .89u(100) + .01u(0)$
- $EU(A') = .11u(100) + .89u(0)$
- $EU(B') = .1u(500) + .9u(0)$

Common Consequence Problem

- Suppose you choose $A \succeq B$
- Then expected utility theory says you *must* choose $A' \succeq B'$

$$
\begin{aligned}
& EU(A') > EU(B') \\
& \iff .11u(100) + .89u(0) \geq .1u(500) + .9u(0) \\
& \iff .11u(100) + .89u(0) \geq .1u(500) + .89u(0) + .01u(0) \\
& \iff .11u(100) + .89u(100) \geq .1u(500) + .89u(100) + .01u(0) \\
& \iff u(100) \geq .1u(500) + .89u(100) + .01u(0) \\
& \iff EU(A) > EU(B)
\end{aligned}
$$

- Typical choice pattern is incompatible with expected utility theory
- Called *common consequence* version of the Allais Paradox, because I added the .89 chance of \$100 million to both sides

The Allais Paradox: Version 2

1. Choose your preferred option:
   C: Receive \$100 million for certain
   D: 98% chance of \$500 million, 2% chance of no money
2. Choose your preferred option:
   C': 1% chance of \$100 million, 99% chance of no money
   D': 0.98% chance of \$500 million, 99.02% chance of no money

- Typical choice pattern? $C \succeq D$; $D' \succeq C'$

- $EU(C) = u(100)$
- $EU(D) = .98u(500) + .02u(0)$
- $EU(C') = .01u(100) + .99u(0)$
- $EU(D') = .098u(500) + .992u(0)$

Common Ratio Problem

- Suppose we observe $C \succeq D$
- Then expected utility theory says we *must* have $C' \succeq D'$

$$
\begin{aligned}
&EU(C) > EU(D) \\
&\iff u(100) \geq .98u(500) + .02u(0) \\
&\iff 0.01u(100) \geq .0098u(500) + .0002u(0) \\
&\iff 0.01u(100) + 0.99u(0) \geq .0098u(500) + .0002u(0) + 0.99u(0) \\
&\iff 0.01u(100) + 0.99u(0) \geq .0098u(500) + .9902u(0) \\
&\iff EU(C') > EU(D')
\end{aligned}
$$

- Called *common ratio* version of the Allais Paradox, because I multipled both sides of the equation by 0.01

What Is Going On?

- Expected utility theory says we should have $A \succeq B \iff A' \succeq B'$ and $C \succeq D \iff C' \succeq D'$
- So if actual behavior doesn't follow these results, expected utility theory must not represent people's true preferences?
- Next time we will see a theory that does explain these choice patterns better