Econ 301: Microeconomic Analysis

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Expected Utility

Motivating Example: Insurance

- Income is \$35,000
- With probability $p = .01$, lose \$10,000 to a house fire
- Can buy \$10,000 of insurance coverage for \$100
  - Then net income will be \$34,900 regardless of whether house fire happens or not
- Which option would you rather have?
  1. 99% chance of \$35,000 with 1% chance of \$25,000
  2. \$34,900 for sure
- Consumer will pick option with higher *expected utility*

Contingent Consumption

- Different *states of the world* with corresponding probabilities
- *Contingent consumption plan*: what consumption will be in each state of the world
- For insurance example:
  - Two states of the world: good (no fire) and bad (fire)
  - Bad state occurs with probability π
  - Income M received in either state
  - Loss L if bad state
  - Choice amount of insurance coverage K
  - Insurance premium γ: cost of getting \$1 of coverage
  - Contigent consumption plan:

$$\begin{array}{llll} M-\gamma K & & = C_g & \text{with probability } 1-\pi \\ M-\gamma K & -L+K & = C_b & \text{with probability } \pi \end{array}$$

Budget Constraint

- Think of consumption in state 1 as a good and consumption in state 2 as another good
- What is formula for budget constraint?
  - Note that $K = \frac{C_b - M + L}{1-\gamma}$ from formula for C_b
  - Plug into formula for C_g we get

$$C_g = M - \gamma \frac{C_b - M + L}{1 - \gamma}$$

- What is slope?
  - Slope = $\frac{-\gamma}{1-\gamma}$
- What point does budget line go through?
  - Goes through net endowment $(M - L, M)$ (ie when $K = 0$)

Budget Constraint Graphically

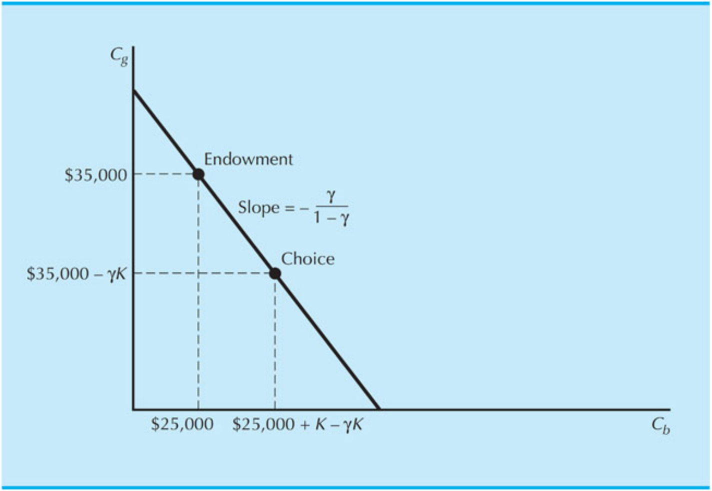

Figure 12.1

Expected Utility

- Consider a general contingent consumption plan

$$A = (\pi_i, c_i)_{i=1}^N = (\pi_1, c_1; \pi_2, c_2; \ldots; \pi_N, c_N)$$

  meaning
  - consume c_1 in state 1, which occurs with probability π_1
  - consume c_2 in state 2, which occurs with probability π_2
  - and so on
- A is also called a *gamble*
- The *expected utility* of A is

$$EU(A) = \sum_i \pi_i u(c_i) = \pi_1 u(c_1) + \pi_2 u(c_2) + \ldots + \pi_N u(c_N)$$

- Compare to the *expected value* of A:

$$EV(A) = \sum_i \pi_i c_i = \pi_1 c_1 + \pi_2 c_2 + \ldots + \pi_N c_N$$

Why this formula?

What Shape Should $u(x)$ Have?

- Consider the following game: I will flip a coin until the first heads comes up. If the first heads is on flip number n, then I'll pay you \$$2^n$. How much would you pay to play this game?
  - Originally proposed by Bernoulli (1738, reprinted 1954)
- What is the expected payoff of this game? Infinite
  - $EV = \frac{1}{2}2 + \frac{1}{4}4 + \frac{1}{8}8 + \ldots = 1 + 1 + 1 + \ldots = \infty$
- It is clear that there must be *diminishing marginal utility of money*
  - Intuition: an extra \$1000 is massive windfall for a very poor person but not even noticeable for very rich person
- We can rationalize the typically observed behavior by assuming that $u(x)$ is concave

Risk Aversion

- If $u(x)$ is concave, we say the underlying preferences are *risk averse*
  - Recall concavity of u means $u'' < 0$
- If risk averse, then $EU(A) < u(EV(A))$ because of concavity of $u(\cdot)$
  - In words: expected utility of a gamble is less than the utility of its expected value
- Useful tip for drawing EU: If gamble A pays off either x_1 or x_2, then $EU(A)$ lies on the line connecting $u(x_1)$ and $u(x_2)$, directly above $EV(A)$

Risk Aversion Graphically

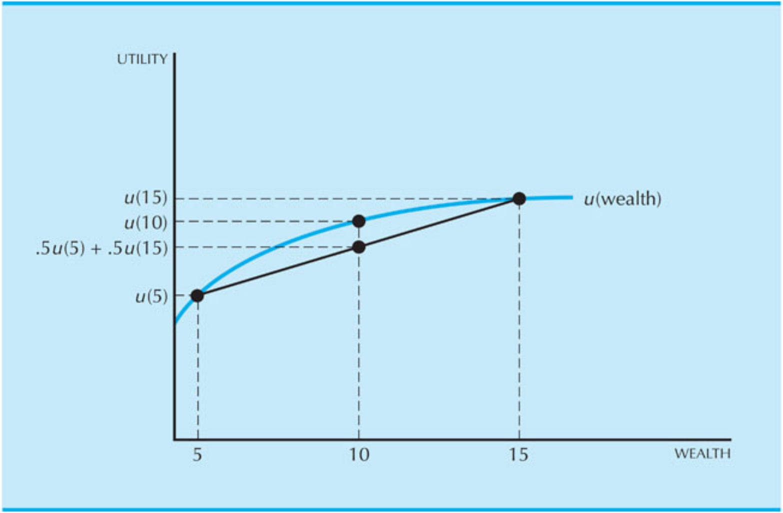

Figure 12.2

Certainty Equivalent and Risk Premium

- The *certainty equivalent* of a gamble A is the amount CE such that $u(CE) = EU(A)$
  - That is, certain amount that gives same utility as uncertain gamble
  - How does certainty equivalent relate to expected value of gamble?
    - For risk averse preference, $CE < EV(A)$
- The *risk premium* is the amount $RP = EV(A) - CE$
  - That is, difference between expected value of gamble and certainty equivalent of gamble
  - What is sign of risk premium?
    - Positive for risk averse preferences

Risk Aversion vs Risk Seeking

- Can also have *risk-seeking* preferences (convex $u(x)$) where all of the above statements are reversed
- Can also have *risk-neutral* preferences (linear $u(x)$)

In summary:

Risk Averse	**Risk Neutral**	**Risk Seeking**
$u(x)$ concave	$u(x)$ linear	$u(x)$ convex
$EU(A) < u(EV(A))$	$EU(A) = u(EV(A))$	$EU(A) > u(EV(A))$
$CE < EV(A)$	$CE = EV(A)$	$CE > EV(A)$
$RP > 0$	$RP = 0$	$RP < 0$

Risk Aversion: Example

- Consider a coin flip for \$15 or \$5
- Let $u(x) = \sqrt{x}$
- Expected value: \$10
- Utility of getting expected value for certain: $U(\$10) = \sqrt{10} = 3.16$
- Expected utility of gamble:

$$EU\left(\frac{1}{2}, \$5; \frac{1}{2}, \$15\right) = \frac{1}{2}\sqrt{5} + \frac{1}{2}\sqrt{15} = \frac{1}{2}2.25 + \frac{1}{2}3.87 = 3.06$$

- Certainty equivalent of gamble: $\sqrt{CE} = 3.06 \rightarrow CE = 3.06^2 = \$9.36$
- Risk premium of gamble: $RP = \$10 - \$9.36 = \$0.64$

Absolute Risk Aversion

- Suppose we want to compare risk aversion across people
- Naively, we may just compare the curvature $u''(x)$
- But this depends on on the scale of utility
- Instead, use the coefficient of *absolute risk aversion*, $-\frac{u''(x)}{u'(x)}$
  - Also know as *Arrow-Pratt measure of risk aversion*
- For risk-averse individual, coefficient must be positive
- Person with higher coefficient is more risk averse

Interpreting Absolute Risk Aversion

- Coefficient may be constant, increasing, or decreasing as x increases
- Constant absolute risk aversion (CARA): as wealth increases, hold same number of dollars in risky asset
- Increasing absolute risk aversion (IARA): as wealth increases, hold fewer dollars in risky asset
- Decreasing absolute risk aversion (DARA): as wealth increases, hold more dollars in risky asset

Relative Risk Aversion

- May want to scale by wealth/income x
- Use *coefficient of relative risk aversion*, $-x\frac{u''(x)}{u'(x)}$
- For risk-averse individual, coefficient must be positive (for positive x)
- Coefficient may be constant (CRRA), increasing (IRRA), or decreasing (DRRA) as x increases
  - Constant relative risk aversion (CRRA): as wealth increases, hold same percentage of dollars in safe asset
  - Increasing relative risk aversion (IRRA): as wealth increases, hold higher percentage of dollars in safe asset
  - Decreasing relative risk aversion (DRRA): as wealth increases, hold lower percentage of dollars in safe asset

Examples

- Does utility function $u(x) = \ln(x)$ exhibit increasing, decreasing, or constant absolute risk aversion?
  - $-\frac{u''(x)}{u'(x)} = -\frac{-\frac{1}{x^2}}{\frac{1}{x}} = \frac{1}{x}$, so decreasing absolute risk aversion
- Does utility function $u(x) = \sqrt{x}$ exhibit increasing, decreasing, or constant absolute risk aversion?
  - $-\frac{u''(x)}{u'(x)} = -\frac{-\frac{1}{4}x^{-3/2}}{\frac{1}{2}x^{-1/2}} = \frac{1}{2x}$, so decreasing absolute risk aversion
- Which one is more risk averse? $u(x) = \ln(x)$ is more risk averse

Returning to the Insurance Example

- What is expected profit for insurance company?
  - If insurance company has to pay out, profit is $\gamma K - K$
  - If not, profit is γK
  - Expected profit of insurance company is $P = \pi(\gamma K - K) + (1 - \pi)\gamma K = (\gamma - \pi)K$
- What should γ equal in a competitive insurance market?
  - Assume company makes no profit, because of competitive pressure from other firms
  - Then $P = 0$, which implies $\gamma = \pi$
  - This is called the *fair insurance price*: eg if there is a 1% chance of disaster, $1 of coverage costs $0.01

Consumer Behavior Under Fair Insurance

- What is expected utility of consumer as function of K? (Assume they have utility function u)

$$EU = (1-\pi)u(M-\gamma K) + \pi u(M-L+(1-\gamma)K)$$

- What is optimal insurance coverage K?
  - From FOC of EU we can get

$$\frac{u'(M-\gamma K)}{u'(M-L+(1-\gamma)K)} = \frac{u'(C_g)}{u'(C_b)} = \frac{\pi}{1-\pi}\frac{1-\gamma}{\gamma}$$

  - Recall that under fair insurance, $\gamma = \pi$
  - Then $\frac{u'(C_g)}{u'(C_b)} = 1$, which implies $C_g = C_b$ or equivalently $K = L$
  - That is, consumer chooses *full insurance* regardless of degree of risk aversion

Appendix

Why Is Expected Utility Reasonable?

- Suppose you make just a few innocuous assumptions about preferences between gambles:
  1. Completeness: For any gambles A and B, either $A \succeq B$ or $B \succeq A$ (or both).
  2. Transitivity: For any gambles A, B, and C, if $A \succeq B$ and $B \succeq C$, then $A \succeq C$.
  3. Continuity: For any gambles A, B, and C, if $A \succeq B \succeq C$ then there exists some number $p \in (0, 1]$ such that $pA + (1 - p)C \sim B$.
  4. Independence: For any gambles A, B, C such that $A \succeq B$ and any $p \in (0, 1]$, we must have $pA + (1 - p)C \succeq pB + (1 - p)C$.

Theorem (von Nueman and Moregensten)

Preferences over gambles that satisfy conditions 1-4 can be represented by expected utility.

The Importance of Independence

- The independence axiom is the most important one for expected utility theory
- What is the intuition for this axiom?
  - How you feel about a prize (ie a specific amount of money) does not depend on the probability you receive it
- How does this manifest in EUT?
  - Let gamble A have three possible outcomes, i.e. $A = (\pi_1, x_1; \pi_2, x_2; \pi_3, x_3)$
  - Recall $EU(A) = \sum_i \pi_i u(x_i) = \pi_1 u(x_1) + \pi_2 u(x_2) + \pi_3 u(x_3)$
  - Note that utility is *additively separable* in probabilities
    - That is, $EU(\pi_1, \pi_2, \pi_3, \cdot) = f_1(\pi_1) + f_1(\pi_2) + f_3(\pi_3)$
  - Note that utility is *linear* in probabilities
    - That is, $EU(\pi_i, \cdot) = a\pi_i + b$ for some constants a and b

Back