

Econ 301: Microeconomic Analysis

Prof. Jeffrey Naecker

Wesleyan University

Game Theory

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Motivation

- ▶ So far in class, have seen only situations where at most one agent can have major impact on outcome
- ▶ Now we turn to case where two or more agents interact *strategically*
- ▶ We need tools from *game theory*

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Setting up the Game

- ▶ A game needs several elements:
 - ▶ *Players*, usually labelled with names, letters, or numbers
 - ▶ *Strategies* for each player
 - ▶ *Payoffs* for each player given a combination of strategies (sometimes called an *outcome*)
- ▶ To start we will analyze *simultaneous move games*, where each player must select strategy without knowing other's strategy

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Payoff Matrix

- ▶ We can handily represent all these elements in a *payoff matrix*
- ▶ For example, if we have the following game:
 - ▶ Players A (row player) and B (column player)
 - ▶ A can choose strategy Top or Bottom
 - ▶ B can choose strategy Left or Right
 - ▶ Payoff function (row player payoff listed first):
 - (Top, Left) → (1, 2)
 - (Top, Right) → (0, 1)
 - (Bottom, Left) → (2, 1)
 - (Bottom, Right) → (1, 0)
- ▶ What is payoff matrix?

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Solution Concepts

- ▶ The game tells us all the possible outcomes
- ▶ A *solution concept* is a rule for narrowing down the possible outcomes
- ▶ Which solution concept we apply depends on the type of game and what we want to assume about the players
- ▶ Solution concepts we will learn:
 - ▶ Dominant/-dominated strategies
 - ▶ Nash equilibrium
 - ▶ Backwards induction/subgame perfect equilibrium

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Best Responses

- ▶ Suppose that the players have the following strategies
 - ▶ (a_1, a_2, \dots, a_n) for player A
 - ▶ (b_1, b_2, \dots, b_m) for player B
- ▶ Define a *best response* for a player as the strategy that maximizes payoffs conditional on a strategy for the other player
 - ▶ A's best response is $BR_A(b)$
 - ▶ B's best response is $BR_B(a)$

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Dominant and Dominated Strategies: Intuition

- ▶ Consider the following game:

		B	
		Left	Right
A	Top	(1, 2)	(0, 1)
	Bottom	(2, 1)	(1, 0)

- ▶ Note that whether B plays Left or Right, A's optimal choice is to choose Bottom (since $2 > 1$ and $1 > 0$)
- ▶ We say that Bottom is a *dominant strategy* for A, and Top is a *dominated strategy*
- ▶ Similarly, for player B, Left is dominant and Right is dominated

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Dominant Strategies

- ▶ The strategy a^D is a *dominant strategy* iff

$$a^D = BR_a(b) \quad \text{for all } b \in b_1, b_2, b_3, \dots$$

- ▶ That is, a^D is *always* the Player A's best response, regardless of what the other player is doing
- ▶ Definition is similar for column player
- ▶ Solution concept: if both players have a dominant strategy, then the game has a *dominant strategy solution* where both players play their dominant strategy

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Dominated Strategies

- ▶ A strategy is *dominated* if it is *never* the best response for a player
 - ▶ (Formal definition is a bit messy)
- ▶ This gives us another solution concept: players will not play dominated strategies
- ▶ Relation to dominant strategies:
 - ▶ Possible to have strategies that are neither dominant nor dominated
 - ▶ In simple 2-by-2 games: if one strategy is dominant, other will be dominated
 - ▶ In more complex games: possible to have strategies that are dominated even if there is no dominant strategy

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Prisoner's Dilemma

- ▶ Consider the following game, called the Prisoner's Dilemma:
 - ▶ Two players are prisoners accused of a joint crime
 - ▶ Can either confess (C) to the crime or deny (D)
 - ▶ Both confess: both get 4 years in jail
 - ▶ Both deny: both get 2 years in jail on lesser charge
 - ▶ One confess and one deny: confessor gets 1 year in jail while denier gets 5 years
- ▶ What is normal form of game?

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Prisoner's Dilemma

- ▶ Does the Prisoner's dilemma have any dominant or dominated strategies?

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Nash Equilibrium: Definition

Definition

A Nash equilibrium of a two-person game is a pair of strategies (a^*, b^*) such that

$$a^* = BR_A(b^*)$$

$$b^* = BR_B(a^*)$$

- ▶ Note that player's actions and beliefs are *mutually consistent*
 - ▶ That is, all players are best responding to each other
- ▶ If all other players are playing NE, no player will want to deviate

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Nash Equilibrium: Another Definition

- ▶ Suppose that more generally we have N players, indexed by i
- ▶ The strategy chosen by player i is noted as s_i
- ▶ The strategy chosen by player i 's opponents is noted as s_{-i}
- ▶ The payoff for player i given her strategy and opponents' strategies is $U_i(s_i, s_{-i})$
 - ▶ Note $BR_i(s_{-i}) = \max_{s_i} U_i(s_i, s_{-i})$
- ▶ Using this notation, we get another definition for Nash equilibrium:

Definition

A Nash equilibrium of a N -person game is a vector of strategies $s_1^*, s_2^*, \dots, s_N^*$ such that

$$U_i(s_i, s_{-i}^*) \geq U_i(s_i, s_{-i}^*)$$

for all s_i and for all $i \in (1, \dots, N)$

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Nash Equilibrium of Prisoner's Dilemma

	<i>Confess</i>	<i>Deny</i>
<i>Confess</i>	(-4, -4)	(-1, -5)
<i>Deny</i>	(-5, -1)	(-2, -2)

- ▶ What is/are the Nash equilibrium/a of the Prisoner's Dilemma?

- ▶ Note that when describing NE, we give strategies, not payoffs

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Another Example: Stag Hunt

	<i>Stag</i>	<i>Hare</i>
<i>Stag</i>	(7, 7)	(0, 1)
<i>Hare</i>	(1, 0)	(2, 2)

- ▶ Are there any dominant or dominated strategies?
- ▶ Are there any NE?

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Mixed Strategies

	<i>L</i>	<i>R</i>
<i>T</i>	(0, 0)	(0, -1)
<i>B</i>	(1, 0)	(-1, 3)

- ▶ This game has no Nash equilibrium in *pure strategies*
- ▶ However, we have not yet considered *mixed strategies*
 - ▶ Players may randomize between two or more strategies
 - ▶ For example, player A could play 50% T, 50% B while player B could play 33% L, 67% R
- ▶ If we allow for mixed strategies, then every game has at least one Nash Equilibrium

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Mixed Strategies: Definition

- ▶ Suppose a player has *pure strategies* (a_1, a_2, \dots, a_n)
- ▶ We then define a *mixed strategy* as a vector $p = (p_1, p_2, \dots, p_n)$ s.t.
 - ▶ $p_i \geq 0$ for all i
 - ▶ $\sum_{i=1}^n p_i = 1$
- ▶ Interpretation: p_i is probability of playing a_i
- ▶ Best response: given mixed strategy for opponent and return the optimal mixed strategy for the player
- ▶ We can modify our definition of Nash to include mixed strategies:

Definition

A Nash Equilibrium of a two-person game is a pair of mixed strategies (p^*, q^*) such that

$$p^* = BR_A(q^*)$$

$$q^* = BR_B(p^*)$$

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Finding Mixed Strategy Solutions

- ▶ Suppose Player A is mixing between strategies Top and Bottom in equilibrium
- ▶ Suppose that playing Top gives greater expected payoff than playing Bottom
- ▶ Then mixing cannot be a best response, since would do better to play pure strategy Top
- ▶ By similar logic, playing Bottom cannot give higher payoff than playing Top
- ▶ Therefore, if a player is mixing in equilibrium, she must be indifferent between all pure strategies she is mixing over

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Finding Mixed Strategies: Example

	<i>Left</i>	<i>Right</i>
<i>Top</i>	(2, 1)	(0, 0)
<i>Bottom</i>	(0, 0)	(1, 2)

- ▶ Suppose players are playing mixed strategies:
 - ▶ Player A is putting weight p on Top, $1 - p$ on Bottom
 - ▶ Player B is putting weight q on Left and $1 - q$ on Right
 - ▶ Player A plays Top: payoff is
 - ▶ Player A plays Bot: payoff is
 - ▶ Player A must be indifferent for mixing:
- ▶ Note that A's indifference condition determines B's mixing probability!

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Finding Mixed Strategies: Example (cont)

	<i>Left</i>	<i>Right</i>
<i>Top</i>	(2, 1)	(0, 0)
<i>Bottom</i>	(0, 0)	(1, 2)

- ▶ Player B plays Left: payoff is
- ▶ Player B plays Right: payoff is
- ▶ Player B must be indifferent for mixing:

- ▶ Thus the mixed strategy Nash equilibrium is
- ▶ Notation: since there are just two strategies for each player, we need just one number to indicate each mixed strategy

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Best Response Functions

- ▶ What is player A's payoff for any mixture q played by B?

- ▶ Thus the best response for A is
 - ▶ Make p as big as possible if $3q - 1 > 0$
 - ▶ Make p as small as possible if $3q - 1 < 0$
 - ▶ Any p is a best response if $3q - 1 = 0$
- ▶ Best response function:

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Best Response Functions (cont)

- ▶ Similarly, we can write the best response function of B as

- ▶ BR functions can also be deduced from indifference conditions
- ▶ Graph q vs p for both players: Any place where the best response functions intersect is a Nash equilibrium

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Best Responses Graphically

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Sequential Games

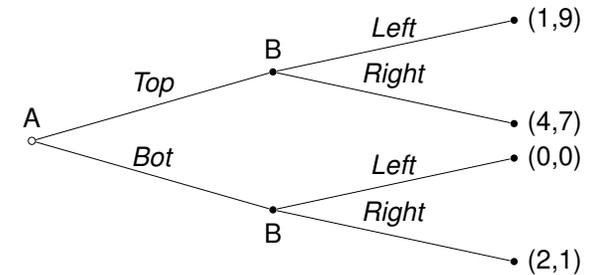
- ▶ Consider the following game
 - ▶ Player A chooses Top or Bottom
 - ▶ Observing A's choice, player B then chooses Left or Right
- ▶ This is a *sequential game*, because players move in sequence rather than simultaneously
- ▶ Payoff function:

(Top, Left)	→	(1, 9)
(Top, Right)	→	(4, 7)
(Bottom, Left)	→	(0, 0)
(Bottom, Right)	→	(2, 1)
- ▶ Note Player B really now has more complicated strategies, since must pick what to do after each move player B

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Extensive Form

- ▶ We analyze such games in *extensive form* with a game tree:



- ▶ Note that extensive form has:
 - ▶ Every non-terminal node labeled with player who moves at that point
 - ▶ Every terminal node labeled with payoffs
 - ▶ Every branch labeled with available actions

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Solution Concept: Subgame Perfect Nash Equilibrium

- ▶ We solve extensive form games with *backwards induction*
 - ▶ Start with end of the game tree
 - ▶ Determine what last mover will do
 - ▶ Take one step backwards in tree and repeat until all decisions have been analyzed
- ▶ The solution we arrive at is called the *subgame perfect Nash equilibrium*
- ▶ Note that in sequential games, strategies must list action at every node at which the player moves
 - ▶ For example, player B's strategy must indicate what B will do if A plays Top *and* what B will do if A plays Bottom
 - ▶ Notation: *RL* means play Right if Top, Left if Bottom, for example

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Example

- ▶ What is backwards induction solution to game on previous slide?

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