Econ 301: Microeconomic Analysis

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Profit Maximization

Motivation

- Technology tells us which combinations of input and outputs are *possible*
- So how do firms pick which set of inputs to actually use?
- They maximize profits
- Assumption throughout this section: firms are price-takers both for selling their output and buying their inputs

Profits

- Setup:
  - Firm output y at price p
  - Firm inputs x_1 and x_2 at prices w_1 and w_2
- Revenue: py
- Cost: $w_1 x_1 + w_2 x_2$
- Profit = revenue - cost:

$$\pi = py - w_1 x_1 - w_2 x_2$$

- Profits and costs typically measured in flows, e.g. wage per month

Opportunity Costs

- Need to be careful to fully capture all inputs
  - If you are self-employed, labor is still an input
  - Price is implicit: what you could get if you worked for someone else
  - Same with rental rate of land, buildings, capital
- In short, opportunity costs are still costs

Profit Maximization

- The firm's profit maximization problem is

$$\max_{x_1, x_2} pf(x_1, x_2) - w_1 x_1 - w_2 x_2$$

- Taking first order conditions, we get

$$MP_1(x_1^*, x_2^*) = \frac{w_1}{p}$$
$$MP_2(x_1^*, x_2^*) = \frac{w_2}{p}$$

- Solving these for x_1^* and x_2^* gives *factor demand curves*

$$x_1^*(w_1, w_2, p)$$
$$x_2^*(w_1, w_2, p)$$

Marginal Product Equals Marginal Cost

- Note that FOC for input i is $MP_i = \frac{w_i}{p}$
- Note that $\frac{w_i}{p}$ is the marginal cost of input i (in terms of output good)
- Thus we can re-state FOC as $MP_i = MC_i$,
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- You may have heard marginal *revenue* equals marginal cost
  - Note that marginal revenue $MR_i = pMP_i$, so our FOC can also be stated as $MR_i = w_i$
  - This is the same statement as above but in dollars instead of product units

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  - Additional revenue: $pMP_1 \Delta x_1$
  - Effect on profits: $\Delta\pi = pMP_1 \Delta x_1 - w_1 \Delta x_1 = (pMP_1 - w_1)\Delta x_1 > 0$
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- Can firm be profit-maximizing if $MP_1 < \frac{w_1}{p}$?
  - Similar argument as above, with $\Delta x_1 < 0$
  - Can raise profits by decreasing use of input 1

Profit Maximization Graphically

- Consider case of just one input, x
- Consider fixed profit $\bar{\pi} = py - wx$
  - We can draw this as $y = \frac{\bar{\pi}}{p} + \frac{w}{p}x$
  - This is *isoprofit curve*

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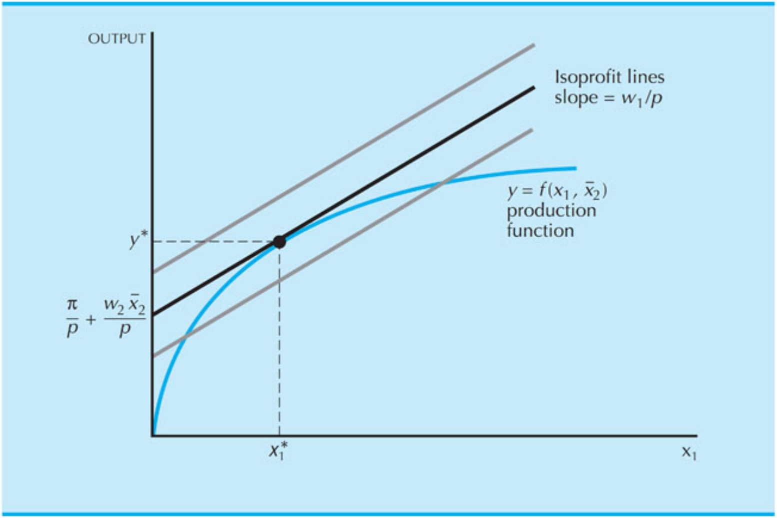

Figure 20.1

Example

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- Set up maximization problem: $\max_{x_1, x_2} p\sqrt{x_1} + \sqrt{x_2} - w_1 x_1 - w_2 x_2$
- Take FOC:

$$p\frac{1}{2}x_1^{-\frac{1}{2}} = w_1 \qquad p\frac{1}{2}x_2^{-\frac{1}{2}} = w_2$$

- Solve for factor demand functions:

$$x_1 = \left(\frac{p}{2w_1}\right)^2 \qquad x_2 = \left(\frac{p}{2w_2}\right)^2$$

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- What is profit?

$$\begin{aligned}
\pi &= py - w_1 x_1 - w_2 x_2 \\
&= p\frac{p}{2}\left(\frac{1}{w_1} + \frac{1}{w_2}\right) - w_1\left(\frac{p}{2w_1}\right)^2 - w_2\left(\frac{p}{2w_2}\right)^2 \\
&= \frac{p^2}{4}\left(\frac{1}{w_1} + \frac{1}{w_2}\right)
\end{aligned}$$

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- What happens to factor demands if w_2 goes up?
  - Firm is not able to change x_2
  - FOC for x_1 does not depend on w_2
  - Therefore firm uses same input to make same amount of output, but profits drop

Returns to Scale

- Firm finds optimal factor demands x_1^*, x_2^*, giving production y^*
- Making profit $\pi^* = py^* - w_1 x_1^* - w_2 x_2^*$
- Assume firm has constant returns to scale
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- So what are only two possibilities?
  - Firm is making zero (or negative) profit
  - Firm is making positive profits but has decreasing returns to scale

Cost Minimization

Motivation

- Alternative way to figure out firm's optimal inputs and outputs: do in two steps
  1. Minimize cost given a level of output
  2. Choose optimal output
- For now, we focus on step one: cost minimization
  - Turns out this step will help us derive supply function

Cost Minimization Problem

- Cost minimization problem is given by

$$\min_{x_1, x_2} w_1 x_1 + w_2 x_2 \text{ s.t. } f(x_1, x_2) = y$$

- In words: choose inputs to find cheapest way to make output equal y
- Note that for this part, we think of y as a constant

Cost Minimization Solution

- Solve by setting up the Lagrangian:

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- Note that we can take ratio of first two FOC:

$$\frac{w_1}{w_2} = \frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}} = \frac{MP_1}{MP_2} = -TRS$$

Cost Function

- Solving for the optimal inputs, we get the *conditional factor demand*:

$$x_1(w_1, w_2, y)$$
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- The formula for the minimized cost is called the *cost function*:

$$c(w_1, w_2, y) = w_1 x_1(w_1, w_2, y) + w_2 x_2(w_1, w_2, y)$$

Cost Minimization Graphically

- Fix cost at some level $c = \bar{c}$
  - We can rearrange cost function $c = w_1 x_1 + w_2 x_2$ to find

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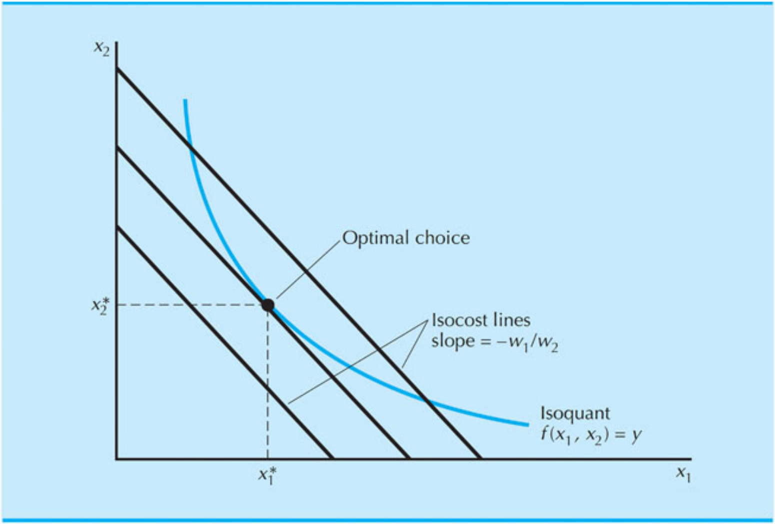

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