Econ 301: Microeconomic Analysis

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Budget Constraints

The Budget Constraint

- Last lecture: what the consumer wants
- This lecture: what they can afford
- The setup
  - Two goods: good 1 and good 2
  - Bundle $X = (x_1, x_2)$
  - Assume consumer also has income m
  - Prices p_1 for good 1 and p_2 for good 2
- Then the *budget constraint* is

$$\underbrace{p_1 x_1}_{\text{money spent on good 1}} + \underbrace{p_2 x_2}_{\text{money spent on good 2}} \leq \underbrace{m}_{\text{money available to spend}}$$

Properties of the Budget Set

- The *budget set* is the set of all bundles such that $p_1 x_1 + p_2 x_2 \leq m$
- The *budget line* is the set of all bundles such that $p_1 x_1 + p_2 x_2 = m$
  - These are the bundles that are just affordable
- What are the slope and intercepts of the budget line?
  - Start with the definition: $p_1 x_1 + p_2 x_2 = m$
  - Solve for x_2 in terms of x_1: $x_2 = \frac{m}{p_2} - \frac{p_1}{p_2} x_1$
  - Slope: $-\frac{p_1}{p_2}$
  - Horizontal intercept: $\frac{m}{p_1}$
  - Vertical intercept: $\frac{m}{p_2}$

Interpreting the Budget Set Properties

- Interpreting the intercepts
  - Tell us how much of a good we get if we spend all of our money on that good
- Interpreting the slope
  1. Tells us the *market substitution rate* of good 1 for good 2
     - Eg if $\frac{p_1}{p_2} = 2$, can buy 2 units of good 2 if you sell one unit of good 1
  2. Also tells us the *opportunity cost* for good 1 in terms of good 2
     - Opportunity cost is the loss of potential gain from other alternatives when one alternative is chosen
     - Eg if $\frac{p_1}{p_2} = 2$, when you buy one additional unit of good 1, you are forgoing two additional units of good 2

Changes in the Budget Set

- What happens when m goes up?
  - A parallel outward shift
- What happens when p_1 goes up?
  - Budget line becomes steeper with vertical intercept unchanged
- What happens when p_1 and p_2 go up by same ratio t?
  - Same as decreasing budget by ratio t
- What happens when m, p_1, and p_2 go up by same ratio t?
  - No change in budget line

How The Government Affects the Budget Set

- Quantity tax: pay amount that depends on the quantity purchased
  - $p \to p + t$, eg $(p_1 + t)x_1 + p_2x_2 = m$
  - Example? gas tax (does not depend on price of a gallon)
- Value tax: pay amount that is percentage of dollars spent on item
  - $p \to (1 + \tau)p$, eg $(1 + \tau)p_1x_1 + p_2x_2 = m$
  - Example? sales tax
- Lump sum: Amount taken from consumer that is independent of any purchasing behavior
  - $m \to m - T$, ie $p_1x_1 + p_2x_2 = m - T$
- Subsidy: the opposite of a tax, so just switch sign
- Rationing: a limit on the amount of a good that can be consumed by any one consumer
  - Effectively "chops off" part of the budget line

Budget Sets and the Real World

1. Two goods are often enough
   - Often we are just interested in the consumption of good 1
   - Let good 2 stand for dollars spent on all other goods: the *composite good*
   - Since this is already in dollar terms, $p_2 = 1$
   - Budget line in this case: $p_1 x_1 + x_2 = m$
2. The numeraire good
   - Similarly, sometimes we are interested in prices and incomes relative to a certain good, say good 2
   - We call this the *numeraire good*
   - Rewrite budget line as: $\frac{p_1}{p_2} x_1 + x_2 = \frac{m}{p_2}$
     - Note that this is the exact same budget, just re-arranging formula
     - Prices are now in terms of the numeraire good rather than dollar terms

Choice

Putting Preferences and Budgets Together

- Preferences and utility: what consumers want
- Budgets: what is available or affordable
- These two concepts combine to tell use what people actually consume
- Guiding principle: consumers choose the best bundle that they can afford

Optimal Choice

- For a given budget set, the consumer should select the bundle in the set that lies on the highest possible indifference curve
  - Anything above this indifference curve is preferred but not affordable
- Hew to we find the location of this bundle in $x_1 - x_2$ space?
- Note that for well-behaved indifference curves the budget line is tangent to the indifference curve
- In general, does this tangency condition have to hold for a bundle to be optimal? No, two counter-examples:
  1. Kinked indifference curves (ie non-differentiable utility function)
  2. Boundary optima (ie exterior solutions to maximization problem)
- However, if we assume no kinked indifference curves and interior solutions, then the tangency condition is *necessary* for optimality

More on necessary and sufficient conditions

Tangency Condition Visualized

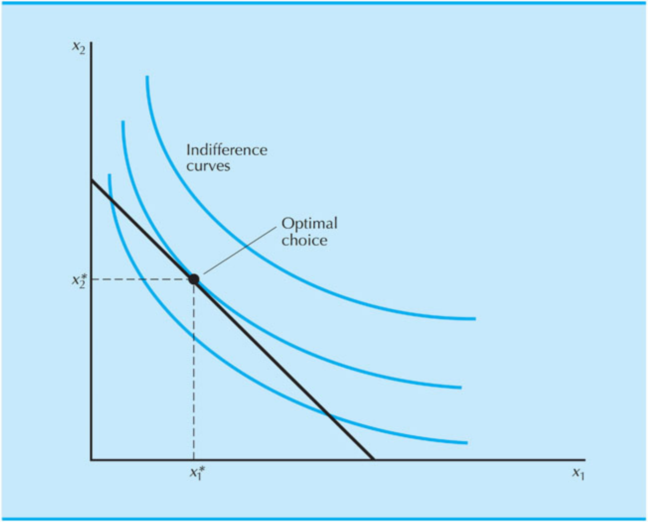


Figure 5.1

Kinked Indifference Curve (Optimal, Not Tangent)

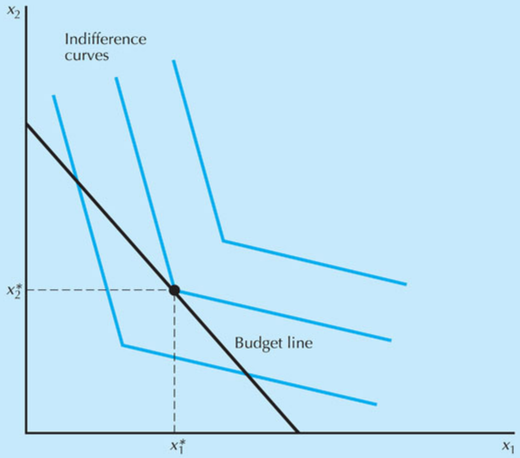
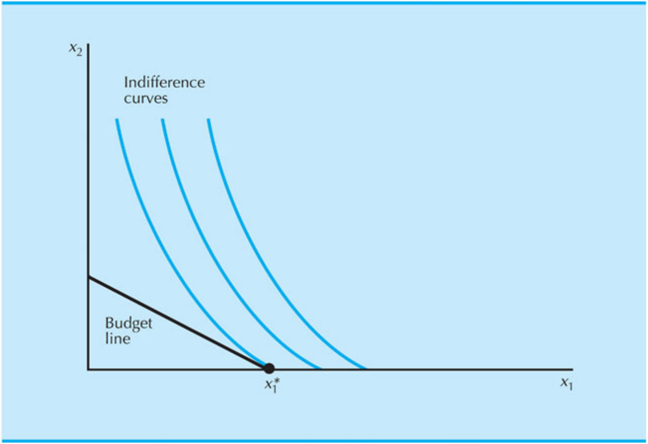


Figure
5.2

Boundary Optimum (Optimal, Not Tangent)

Figure 5.3



When Is Tangency Equivalent to Optimality?

- We just showed that if the utility function is differentiable (no kinks in indifference curves) and solutions are interior, then tangency is necessary for optimality (ie optimality implies tangency)
- What about the other way around? Does tangency imply optimality in general? No. Consider non-convex preferences:
  - If preferences are not convex, a tangency point might not be the optimal choice
- However, if we have convexity, no kinks, and interior solutions, we are OK to assume that tangency is equivalent to optimality

Theorem

If preferences are convex, the utility function is differentiable, and we consider only interior solutions, then tangency of the budget set and the indifference curve is necessary and sufficient for (ie equivalent to) optimality.

Non-Convex Preferences (Tangent, Not Optimal)

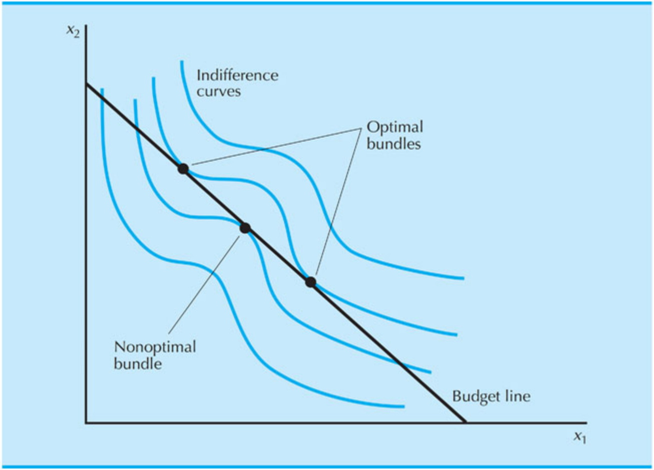


Figure 5.4

The Tangency Condition

- The slope of the budget line is $-\frac{p_1}{p_2}$
- The slope of the indifference curve is $MRS = -\frac{MU_1}{MU_2}$
- So we get the tangency condition:

$$MRS = -\frac{p_1}{p_2}$$

or equivalently

$$\frac{MU_1}{MU_2} = \frac{p_1}{p_2}$$

- If this condition did not hold, consumer could trade with market to make herself better off
- Tangency condition plus budget constraint give two equations for two unknowns (optimal x_1 and x_2)

Consumer Demand

- Note that the solution for the optimal bundle depends on prices and income
- The function that relates the optimal bundle to these variables is the *demand* function:

$$X(p_1, p_2, m) = (x_1(p_1, p_2, m), x_2(p_1, p_2, m))$$

- Often we write (x_1^*, x_2^*) to remind ourselves that this is the optimal bundle
- Formally, demand is the solution to a *constrained optimization* problem:

$$(x_1^*, x_2^*) = \arg\max_{x_1, x_2} u(x_1, x_2) \text{ s.t. } p_1 x_1 + p_2 x_2 \leq m$$

Demand from the Lagrangian

- Lagrangian is $\mathcal{L}(x_1, x_2, \lambda) = u(x_1, x_2) + \lambda(m - p_1 x_1 - p_2 x_2)$
- First order conditions:

$$0 = \frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial u}{\partial x_1} - \lambda p_1 \qquad (1)$$

$$0 = \frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial u}{\partial x_2} - \lambda p_2 \qquad (2)$$

$$0 = \frac{\partial \mathcal{L}}{\partial \lambda} = m - p_1 x_1 - p_2 x_2 \qquad (3)$$

- Rearrange the first two equations and take their ratio to derive the tangency condition:

$$\frac{p_1}{p_2} = \frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \frac{MU_1}{MU_2}$$

Can also do with substitution of constraint

Examples

Cobb-Douglas

- Assume $u(x_1, x_2) = x_1^c x_2^d$. What is demand?
- Set up Lagrangian:

$$\mathcal{L} = x_1^c x_2^d + \lambda(m - p_1 x_1 - p_2 x_2)$$

- Take FOC:

$$\begin{aligned} x_1 : &\quad c x_1^{c-1} x_2^d - \lambda p_1 = 0 \\ x_2 : &\quad d x_1^c x_2^{d-1} - \lambda p_2 = 0 \\ \lambda : &\quad m - p_1 x_1 - p_2 x_2 = 0 \end{aligned}$$

- Take ratio of first two FOC:

$$\frac{c x_1^{c-1} x_2^d}{d x_1^c x_2^{d-1}} = \frac{\lambda p_1}{\lambda p_2} \to \frac{c}{d} p_2 x_2 = p_1 x_1$$

Cobb-Douglas con't

- Plug in to third FOC (budget constraint):

$$\frac{c}{d}p_2x_2 + p_2x_2 = m$$

- Solve for x_2 and then x_1:

$$\begin{aligned} x_1^* &= \tfrac{c}{c+d}\tfrac{m}{p_1} \\ x_2^* &= \tfrac{d}{c+d}\tfrac{m}{p_2} \end{aligned}$$

- Note by rearranging we get

$$\begin{aligned} \frac{p_1x_1^*}{m} &= \tfrac{c}{c+d} \\ \frac{p_2x_2^*}{m} &= \tfrac{d}{c+d} \end{aligned}$$

- So fraction of income spent on each good is a constant

Perfect Substitutes

- What is the formula for demand for perfect substitutes in a 1-to-1 ratio?
- Hint: our assumption of interior solution will not apply, so need to use graphical arguments
- Solution:

$$x_1^* \begin{cases} = 0 & \text{if } p_1 > p_2 \\ \in \left[0, \frac{m}{p_1}\right] & \text{if } p_1 = p_2 \\ = \frac{m}{p_1} & \text{if } p_1 < p_2 \end{cases}$$

$$x_2^* = \frac{m}{p_2} - \frac{p_1}{p_2} x_1$$

- Intuition: consume as much as possible of cheaper good

Perfect Complements

- What is the formula for demand for perfect complements in a 1-to-1 ratio?
- Hint: our assumption of no kinks will not apply, so again need to reason without calculus
- Solution:

$$
\begin{aligned}
x_1^* &= \frac{m}{p_1+p_2} \\
x_2^* &= \frac{m}{p_1+p_2}
\end{aligned}
$$

- Intuition: Consumer is really interested in being pairs of goods, so price is effectively sum of individual prices

Appendix

Sidebar: Necessary and Sufficient Conditions

- Suppose we have the following logical relation between two conditions X and Y:

$$X \implies Y$$

  - Meaning: if X holds, then Y will hold as well (though not necessarily the other way around
  - Read "X implies Y"
- We say that Y is *necessary* for X; that is, Y necessarily holds if X holds
- We say that X is *sufficient* for Y; that is, X occurring is sufficient for us to know that Y will hold as well
- If X is both necessary and sufficient for Y, then the two statements are logically equivalent
  - Written $X \iff Y$
  - Read "X if and only if Y"

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Solving The Optimization Problem: Substitution Method

- We want to solve

$$\max_{x_1, x_2} u(x_1, x_2) \text{ s.t. } p_1 x_1 + p_2 x_2 \leq m$$

- First, solve constraint for x_2 in terms of x_1: $x_2 = \frac{m}{p_2} - \frac{p_1}{p_2} x_1$
- Substitution leads to an *unconstrained* maximization problem:

$$\max_{x_1} u(x_1, x_2(x_1)) = \max_{x_1} u\left(x_1, \frac{m}{p_2} - \frac{p_1}{p_2} x_1\right)$$

- Take the *first order condition*, ie set the derivative w.r.t the optimizing variable equal to zero:

$$0 = \frac{du}{dx_1} = \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} \frac{dx_2}{dx_1}$$

where we have used the version of the chain rule for multiple variables

Substitution Method Continued

- Rearranging:
$$\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = -\frac{dx_2}{dx_1}$$
- From $x_2 = \frac{m}{p_2} - \frac{p_1}{p_2}x_1$:
$$\frac{dx_2}{dx_1} = -\frac{p_1}{p_2}$$
- Combining we recover the tangency condition:
$$\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \frac{p_1}{p_2}$$

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