Econ 301: Microeconomic Analysis

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Game Theory

Motivation

- So far in class, have seen only situations where at most one agent can have major impact on outcome
- Now we turn to case where two or more agents interact *strategically*
- We need tools from *game theory*

Setting up the Game

- A game needs several elements:
  - *Players*, usually labelled with names, letters, or numbers
  - *Strategies* for each player
  - *Payoffs* for each player given a combination of strategies (sometimes called an *outcome*)
- To start we will analyze *simultaneous move games*, where each player must select strategy without knowing other's strategy

Payoff Matrix

- We can handily represent all these elements in a *payoff matrix*
- For example, if we have the following game:
  - Players A (row player) and B (column player)
  - A can choose strategy Top or Bottom
  - B can choose strategy Left or Right
  - Payoff function (row player payoff listed first):

    (Top, Left) $\rightarrow (1, 2)$
    (Top, Right) $\rightarrow (0, 1)$
    (Bottom, Left) $\rightarrow (2, 1)$
    (Bottom, Right) $\rightarrow (1, 0)$
- What is payoff matrix?

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		B	
		Left	*Right*
A	*Top*	$(1, 2)$	$(0, 1)$
	Bottom	$(2, 1)$	$(1, 0)$

Solution Concepts

- The game tells us all the possible outcomes
- A *solution concept* is a rule for narrowing down the possible outcomes
- Which solution concept we apply depends on the type of game and what we want to assume about the players
- Solution concepts we will learn:
    - Dominant/dominated strategies
    - Nash equilibrium
    - Backwards induction/subgame perfect equilibrium

Best Responses

- Suppose that the players have the following strategies
  - $(a_1, a_2, \ldots, a_n)$ for player A
  - $(b_1, b_2, \ldots, b_m)$ for player B
- Define a *best response* for a player as the strategy that maximizes payoffs conditional on a strategy for the other player
  - A's best response is $BR_A(b)$
  - B's best response is $BR_B(a)$

Dominant and Dominated Strategies: Intuition

- Consider the following game:

		B	
		Left	*Right*
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	Bottom	$(\underline{2}, 1)$	$(\underline{1}, 0)$

- Note that whether B plays Left or Right, A's optimal choice is to choose Bottom (since $2 > 1$ and $1 > 0$)
- We say that Bottom is a *dominant strategy* for A, and Top is a *dominated strategy*

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- We say that Bottom is a *dominant strategy* for A, and Top is a *dominated strategy*
- Similarly, for player B, Left is dominant and Right is dominated

Dominant Strategies

- The strategy a^D is a *dominant strategy* iff

$$a^D = BR_a(b) \quad \text{for all } b \in b_1, b_2, b_3, \ldots.$$

- That is, a^D is *always* the Player A's best response, regardless of what the other player is doing
- Definition is similar for column player
- Solution concept: if both players have a dominant strategy, then the game has a *dominant strategy solution* where both players play their dominant strategy

Dominated Strategies

- A strategy is *dominated* if it is *never* the best response for a player
  - (Formal definition is a bit messy)
- This gives us another solution concept: players will not play dominated strategies
- Relation to dominant strategies:
  - Possible to have strategies that are neither dominant nor dominated
  - In simple 2-by-2 games: if one strategy is dominant, other will be dominated
  - In more complex games: possible to have strategies that are dominated even if there is no dominant strategy

Prisoner's Dilemma

- Consider the following game, called the Prisoner's Dilemma:
  - Two players are prisoners accused of a joint crime
  - Can either confess (C) to the crime or deny (D)
  - Both confess: both get 4 years in jail
  - Both deny: both get 2 years in jail on lesser charge
  - One confess and one deny: confessor gets 1 year in jail while denier gets 5 years

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	Confess	*Deny*
Confess	$(-4, -4)$	$(-1, -5)$
Deny	$(-5, -1)$	$(-2, -2)$

Prisoner's Dilemma

- Does the Prisoner's dilemma have any dominant or dominated strategies?

Prisoner’s Dilemma

- Does the Prisoner’s dilemma have any dominant or dominated strategies?
  - Confess is a dominant strategy for both players
  - Deny is a dominated strategy for both players
  - Thus the only possible outcome according to both solution concepts is (Confess, Confess)

Nash Equilibrium: Definition

Definition

A Nash equilibrium of a two-person game is a pair of strategies (a^*, b^*) such that

$$a^* = BR_A(b^*)$$
$$b^* = BR_B(a^*)$$

- Note that player's actions and beliefs are *mutually consistent*
  - That is, all players are best responding to each other
- If all other players are playing NE, no player will want to deviate

Nash Equilibrium: Another Definition

- Suppose that more generally we have N players, indexed by i
- The strategy chosen by player i is noted as s_i
- The strategy chosen by player i's opponents is noted as s_{-i}
- The payoff for player i given her strategy and opponents' strategies is $U_i(s_i, s_{-i})$
  - Note $BR_i(s_{-i}) = \max_{s_i} U_i(s_i, s_{-i})$

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  - Note $BR_i(s_{-i}) = \max_{s_i} U_i(s_i, s_{-i})$
- Using this notation, we get another definition for Nash equilibrium:

Definition

A Nash equilibrium of a N-person game is a vector of strategies $s_1^*, s_2^*, \ldots, s_N^*$ such that

$$U_i(s_i,^* s_{-i}^*) \geq U_i(s_i, s_{-i}^*)$$

for all s_i and for all $i \in (1, \ldots, N)$

Nash Equilibrium of Prisoner's Dilemma

	Confess	*Deny*
Confess	$(-4, -4)$	$(-1, -5)$
Deny	$(-5, -1)$	$(-2, -2)$

- What is/are the Nash equilibrium/a of the Prisoner's Dilemma?

Nash Equilibrium of Prisoner's Dilemma

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- What is/are the Nash equilibrium/a of the Prisoner's Dilemma?
  - If your opponent is choosing Deny, your best response to choose Confess, since $(-1 > -2)$
  - If your opponent is choosing Confess, your best response to choose Confess, since $(-4 > -5)$
  - Thus NE is (Confess, Confess)
  - Note that NE occurs whenever both entries in a cell are best responses
- Note that when describing NE, we give strategies, not payoffs

Another Example: Stag Hunt

	Stag	*Hare*
Stag	$(7, 7)$	$(0, 1)$
Hare	$(1, 0)$	$(2, 2)$

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- Are there any dominant or dominated strategies?
  - No dominant or dominated strategies for either player in the above game
- Are there any NE?
  - Note that for outcome (Stag, Stag)
    - A is choosing optimal response to B's strategy
    - B is choosing optimal response to A's strategy
  - We call an outcome with mutual best response a *Nash equilibrium*
  - (Hare,Hare) is a Nash equilibrium as well

Mixed Strategies

	L	R
T	$(0, \underline{0})$	$(\underline{0}, -1)$
B	$(\underline{1}, 0)$	$(-1, \underline{3})$

- This game has no Nash equilibrium in *pure strategies*
- However, we have not yet considered *mixed strategies*
  - Players may randomize between two or more strategies
  - For example, player A could play 50% T, 50% B while player B could play 33% L, 67% R
- If we allow for mixed strategies, then every game has at least one Nash Equilbrium

Mixed Strategies: Definition

- Suppose a player has *pure* strategies $(a_1, a_2, \ldots, a_n)$
- We then define a *mixed strategy* as a vector $p = (p_1, p_2, \ldots, p_n)$ s.t.
  - $p_i \geq 0$ for all i
  - $\sum_{i=1}^{n} p_i = 1$
- Interpretation: p_i is probability of playing a_i

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- Interpretation: p_i is probability of playing a_i
- Best response: given mixed strategy for opponent and return the optimal mixed strategey for the player
- We can modify our definition of Nash to include mixed strategies:

Definition

A Nash Equilibrium of a two-person game is a pair of mixed strategies (p^*, q^*) such that

$$p^* = BR_A(q^*)$$
$$q^* = BR_B(p^*)$$

Finding Mixed Strategy Solutions

- Suppose Player A is mixing between strategies Top and Bottom in equilibrium
- Suppose that playing Top gives greater expected payoff than playing Bottom
- Then mixing cannot be a best response, since would do better to play pure strategy Top

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- Then mixing cannot be a best response, since would do better to play pure strategy Top
- By similar logic, playing Bottom cannot give higher payoff than playing Top
- Therefore, if a player is mixing in equilibrium, she must be indifferent between all pure strategies she is mixing over

Finding Mixed Strategies: Example

	Left	*Right*
Top	$(2, 1)$	$(0, 0)$
Bottom	$(0, 0)$	$(1, 2)$

- Suppose players are playing mixed strategies:
  - Player A is putting weight p on Top, $1 - p$ on Bottom
  - Player B is putting weight q on Left and $1 - q$ on Right

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- Player A must be indifferent for mixing:

$$2q = 1 - q \rightarrow q = \frac{1}{3}$$

- Note that A's indifference condition determines B's mixing probability!

Finding Mixed Strategies: Example (cont)

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Top	$(2, 1)$	$(0, 0)$
Bottom	$(0, 0)$	$(1, 2)$

- Player B plays Left: payoff is

Finding Mixed Strategies: Example (cont)

	Left	*Right*
Top	$(2, 1)$	$(0, 0)$
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- Player B plays Left: payoff is $1p + 0(1 - p) = p$

Finding Mixed Strategies: Example (cont)

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- Player B plays Right: payoff is

Finding Mixed Strategies: Example (cont)

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- Player B plays Right: payoff is $0p + 2(1 - p) = 2 - 2p$

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- Thus the mixed strategy Nash equilibrium is $(\frac{2}{3}, \frac{1}{3})$
- Notation: since there are just two strategies for each player, we need just one number to indicate each mixed strategy

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$$\begin{aligned}\pi_A &= 2pq + 0(1-p)q + 0p(1-q) + 1(1-p)(1-q) \\ &= (3q-1)p + 1 - q\end{aligned}$$

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$$p = BR_A(q) \begin{cases} = 1 \text{ if } q > 1/3 \\ \in [0,1] \text{ if } q = 1/3 \\ = 0 \text{ if } q < 1/3 \end{cases}$$

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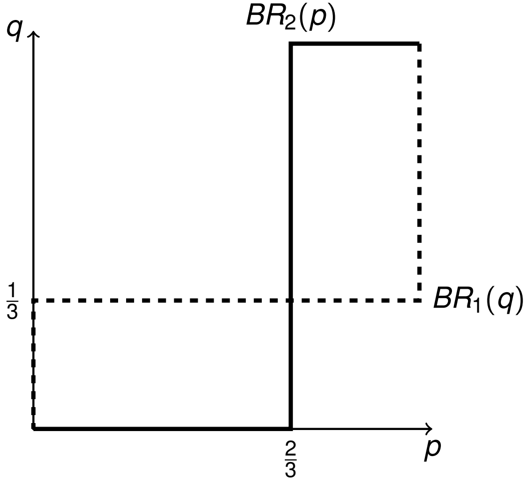
- BR functions can also be deduced from indifference conditions
- Graph q vs p for both players: Any place where the best response functions intersect is a Nash equilibrium

Best Responses Graphically



- Each intersection represents one (pure or mixed) NE

Sequential Games

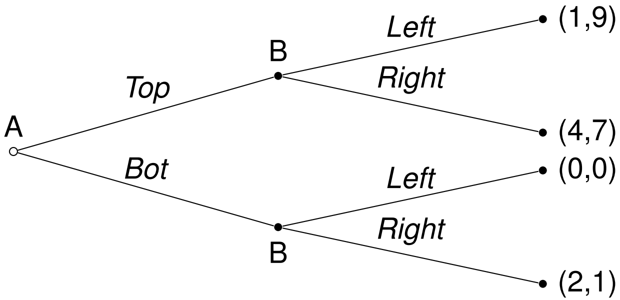
- Consider the following game
  - Player A chooses Top or Bottom
  - Observing A's choice, player B then chooses Left or Right
- This is a *sequential game*, because players move in sequence rather than simultaneously
- Payoff function:

$$\begin{array}{ll} \text{(Top, Left)} & \rightarrow (1, 9) \\ \text{(Top, Right)} & \rightarrow (4, 7) \\ \text{(Bottom, Left)} & \rightarrow (0, 0) \\ \text{(Bottom, Right)} & \rightarrow (2, 1) \end{array}$$

- Note Player B really now has more complicated strategies, since must pick what to do after each move player B

Extensive Form

- We analyze such games in *extensive form* with a game tree:



- Note that extensive form has:
  - Every non-terminal node labeled with player who moves at that point
  - Every terminal node labeled with payoffs
  - Every branch labeled with available actions

Solution Concept: Subgame Perfect Nash Equilibrium

- We solve extensive form games with *backwards induction*
  - Start with end of the game tree
  - Determine what last mover will do
  - Take one step backwards in tree and repeat until all decisions have been analyzed

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- We solve extensive form games with *backwards induction*
  - Start with end of the game tree
  - Determine what last mover will do
  - Take one step backwards in tree and repeat until all decisions have been analyzed
- The solution we arrive at is called the *subgame perfect Nash equilibrium*
- Note that in sequential games, strategies must list action at every node at which the player moves
  - For example, player B's strategy must indicate what B will do if A plays Top *and* what B will do if A plays Top
  - Notation: *RL* means play Right if Top, Left if Bottom, for example

Example

- What is backwards induction solution to game on previous slide?

Example

- What is backwards induction solution to game on previous slide?
  - After Top, player B will play Left
  - After Bottom, player B will play Right
  - Given what player B will do, player A will choose to play Bottom
  - SPNE strategies are (B, LR)
  - SPNE outcome is (Bottom, Right)